SOLUTIONS TO SELECTED QUESTIONS IN HOMEWORK 18

MATH 241

13.4.3

Proof. The two boundary conditions implies the solution has the form $u(x,t) = \sum\limits_{n=1}^{\infty}(A_n \cos\frac{an\pi}{L}t + B_n \sin\frac{an\pi}{L}t)\sin\frac{n\pi}{L}x$, $\frac{\partial u}{\partial t}(x,t) = \sum\limits_{n=1}^{\infty}(-A_n\frac{an\pi}{L}\sin\frac{an\pi}{L}t + B_n\frac{an\pi}{L}\cos\frac{an\pi}{L}t)\sin\frac{n\pi}{L}x$. So the initial condition $\frac{\partial u}{\partial t}|_{t=0} = 0$ implies $B_n = 0$, the initial condition $u(x,0) = f(x)$ implies $\sum\limits_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}x = f(x)$.

$f(x)$ is described piecewisely. When $0 < x < \frac{L}{3}$, it has slope $\frac{1}{\frac{L}{3}} = \frac{3}{L}$, so it is equal to $\frac{3}{L}x$. When $\frac{L}{3} < x < \frac{2L}{3}$, it is 1. When $\frac{2L}{3} < x < L$, it is the segment from $(\frac{2L}{3}, 1)$ to $(L, 0)$, so the equation is $\frac{y-0}{x-L} = \frac{1-0}{\frac{2L}{3}-L} = -\frac{3}{L}$, i.e., $y = -\frac{3}{L}(x-L)$.

So the coefficients of the Fourier sine series for $f(x)$ is

$$\begin{array}{rc}
 & \frac{2}{L}\int_0^{\frac{L}{3}}\frac{3}{L}x\sin\frac{n\pi}{L}x\,dx + \frac{2}{L}\int_{\frac{L}{3}}^{\frac{2L}{3}}\sin\frac{n\pi}{L}x\,dx + \frac{2}{L}\int_{\frac{2L}{3}}^{L}-\frac{3}{L}(x-L)\sin\frac{n\pi}{L}x\,dx \\
= & 6(-\frac{1}{3n\pi}\cos\frac{n\pi}{3} + \frac{1}{(n\pi)^2}\sin\frac{n\pi}{3}) + (-\frac{2}{n\pi}\cos\frac{2n\pi}{3} + \frac{2}{n\pi}\cos\frac{n\pi}{3}) - 6(-\frac{1}{3n\pi}\cos\frac{2n\pi}{3} + \frac{1}{(n\pi)^2}\sin\frac{2n\pi}{3}) \\
= & \frac{6}{(n\pi)^2}(\sin\frac{n\pi}{3} - \sin\frac{2n\pi}{3})
\end{array}$$

Note that $\sin\frac{2n\pi}{3} = \sin(n\pi - \frac{n\pi}{3}) = (-1)^{n+1}\sin\frac{n\pi}{3}$, so the answer above can be written as $\frac{6(1-(-1)^{n+1})}{(n\pi)^2}\sin\frac{n\pi}{3} = \frac{6(1+(-1)^n)}{(n\pi)^2}\sin\frac{n\pi}{3}$. Therefore the answer is

$$u(x,t) = \sum_{n=1}^{\infty}\frac{6(1+(-1)^n)}{(n\pi)^2}\sin\frac{n\pi}{3}\cos\frac{an\pi}{L}t\sin\frac{n\pi}{L}x$$

□

13.4.11

Proof. The solution has the form

$$u(x,t) = \sum_{n=1}^{\infty}(A_n\cos\frac{an\pi}{L}t + B_n\sin\frac{an\pi}{L}t)\sin\frac{n\pi}{L}x$$

$\frac{\partial u}{\partial t}(x,0) = 0$ implies $B_n = 0$, so $u(x,t) = \sum\limits_{n=1}^{\infty} A_n\cos\frac{an\pi}{L}t\sin\frac{n\pi}{L}x = \frac{1}{2}\sum\limits_{n=1}^{\infty} A_n[\sin(\frac{n\pi}{L}x + \frac{an\pi}{L}t) + \sin(\frac{n\pi}{L}x - \frac{an\pi}{L}t)]$, this can be written as

$$u(x,t) = \frac{1}{2}[\sum_{n=1}^{\infty}A_n\sin\frac{n\pi}{L}(x+at) - \sum_{n=1}^{\infty}A_n\sin\frac{n\pi}{L}(x-at)]$$

By the other initial condition,

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L} x$$

So the above solution can be written as $u(x,t) = \frac{1}{2}[f(x+at) - f(x-at)]$. □

13.2.12

Proof. The steady state of temperature is described by harmonic functions. Left end is where $x = 0$, right end is where $x = \pi$, bottom is where $y = 0$. So the BVP is:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < \pi, y > 0$$

$$u(0,y) = e^{-y}, u(\pi, y) = \begin{cases} 100 & \text{if } 0 < y \leq 1 \\ 0 & \text{if } y > 1 \end{cases}$$

$$u(x,0) = f(x)$$

□

13.5.11

Proof. Although this is a finite interval times a semi-infinite interval situation, but the boundary condition is not homogeneous. If you try the Fourier sine transform method, you will find yourself facing a nonhomogeneous second order linear ODE, and it is in general very hard to solve. So for the nonhomogeneous boundary condition for a finite interval times a semi-infinite interval situation, instead of Fourier transform, you would prefer separation of variables.

Let $y = X(x)Y(y)$ again, the boundary condition $u(0,y) = 0, u(\pi, y) = 0$ implies the eigenvalues are $\lambda = n^2$, $n = 1, 2, \cdots$, and the solution has form

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cosh ny + B_n \sinh ny) \sin nx$$

Now notice the condition $u(x,y)$ is bounded at $y \to \infty$, what does that imply?

The part involving y is $A_n \cosh ny + B_n \sinh ny = \frac{A_n + B_n}{2} e^{ny} + \frac{A_n - B_n}{2} e^{-ny}$, and e^{ny} blows up at infinity, while e^{-ny} decays to zero. So the condition $u(x,y)$ is bounded at $y \to \infty$ forces $A_n + B_n = 0$.

This analysis reminds us that, in this semi-infinite situation, with the boundedness condition, it seems that using the e^y, e^{-y} combination is better than using the $\cosh y, \sinh y$ combination. So we will use the form

$$u(x,t) = \sum_{n=1}^{\infty} (C_n e^{ny} + D_n e^{-ny}) \sin nx$$

and we know that C_n must be zero to kill the blowing-up term e^{ny}.

Thus the solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} D_n e^{-ny} \sin nx$$

Let $y = 0$, we get $f(x) = \sum\limits_{n=1}^{\infty} D_n \sin nx$, so $D_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx dx$, and the solution is written as

$$u(x,t) = \sum_{n=1}^{\infty} (\frac{2}{\pi}\int_0^{\pi} f(x) \sin nx dx) e^{-ny} \sin nx$$

□

Remark If the region is $0 < x < \pi, y < 0$, i.e., y goes to infinity in the negative direction, then it is e^{-ny} term that blows up, and so e^{ny} should be left.

13.5.16

Proof. It is a square plate, and $0 < x < 2$, so also $0 < y < 2$.

Decompose the BVP into two. The first BVP is:

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0, 0 < x < 2, 0 < y < 2$$

$$u_1(0,y) = 0, u_1(2,y) = y(2-y)$$

$$u_1(x,0) = 0, u_1(x,2) = 0$$

The solution has the form

$$u_1(x,t) = \sum_{n=1}^{\infty} (A_n \cosh \frac{n\pi}{2} x + B_n \sinh \frac{n\pi}{2} x) \sin \frac{n\pi}{2} y$$

Let $x = 0$ we should get 0, so $A_n = 0$. The coefficients for Fourier sine series of $u(2,y)$ is

$$\int_0^2 y(2-y) \sin \frac{n\pi}{2} y dy = -\frac{8(-1)^n}{n\pi} - \frac{16(1-(-1)^n)}{(n\pi)^3}$$

, so $B_n = -\frac{8(-1)^n}{n\pi \sinh n\pi} - \frac{16(1-(-1)^n)}{(n\pi)^3 \sinh n\pi}$.

So the solution to the first BVP is

$$u_1(x,t) = \sum_{n=1}^{\infty} [-\frac{8(-1)^n}{n\pi \sinh n\pi} - \frac{16(1-(-1)^n)}{(n\pi)^3 \sinh n\pi}] \sinh \frac{n\pi}{2} x \sin \frac{n\pi}{2} y$$

The second BVP is:

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0, 0 < x < 2, 0 < y < 2$$

$$u_2(0,y) = 0, u_2(2,y) = 0$$

$$u_2(x,0) = 0, u_2(x,2) = \begin{cases} x & \text{if } 0 < x < 1 \\ 2-x & \text{if } 1 \le x < 2 \end{cases}$$

The solution has the form

$$u_2(x,t) = \sum_{n=1}^{\infty}(C_n \cosh \frac{n\pi}{2}y + D_n \sinh \frac{n\pi}{2}y) \sin \frac{n\pi}{2}x$$

Let $x = 0$ we should get 0, so $C_n = 0$. The coefficients for Fourier sine series of $u(2,y)$ is

$$\int_0^1 y \sin \frac{n\pi}{2}ydy + \int_1^2 (2-y) \sin \frac{n\pi}{2}ydy = \frac{8}{(n\pi)^2}$$

, so $D_n = \frac{8}{(n\pi)^2 \sinh n\pi}$.

So the solution to the second BVP is

$$u_2(x,t) = \sum_{n=1}^{\infty}[\frac{8}{(n\pi)^2 \sinh n\pi}] \sinh \frac{n\pi}{2}y \sin \frac{n\pi}{2}x$$

In combination, the solution to the original BVP is

$$u_1(x,t)+u_2(x,t) = \sum_{n=1}^{\infty}[-\frac{8(-1)^n}{n\pi \sinh n\pi} - \frac{16(1-(-1)^n)}{(n\pi)^3 \sinh n\pi}] \sinh \frac{n\pi}{2}x \sin \frac{n\pi}{2}y + \sum_{n=1}^{\infty}[\frac{8}{(n\pi)^2 \sinh n\pi}] \sinh \frac{n\pi}{2}y \sin \frac{n\pi}{2}x$$

□

15.4.10

Proof. This is a homogeneous Dirichlet BVP for a semi-infinite interval, so use Fourier sine transform. The equation becomes

$$\frac{\partial^2 \mathcal{F}_s\{u\}}{\partial t^2} + a^2\alpha^2 \mathcal{F}_s\{u\} = 0$$

The solution is $\mathcal{F}_s\{u\} = C_1(\alpha)\cos a\alpha t + C_2(\alpha)\sin a\alpha t$. The boundary condition tells us $C_2(\alpha) = 0$, $C_1(\alpha) = \mathcal{F}_s\{xe^{-x}\} = \frac{2\alpha}{(1+\alpha^2)^2}$. So $\mathcal{F}_s\{u\} = \frac{2\alpha}{(1+\alpha^2)^2}\cos a\alpha t$, and

$$u = \frac{2}{\pi}\int_0^{\infty} \frac{2\alpha}{(1+\alpha^2)^2} \cos a\alpha t \sin \alpha x d\alpha$$

□

15.4.18

Proof. Decompose the BVP into two. The first BVP is:

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0, 0 < x < \pi, y > 0$$

$$u_1(0,y) = 0, u_1(\pi,y) = 0$$

$$u_1(x,0) = f(x)$$

This is a nonhomogeneous boundary condition for the infinite direction, so according to our experience in 13.5.11, we should use separation of variable method, and by boundedness we know the solution has the form

$$u_1(x,t) = \sum_{n=1}^{\infty} C_n e^{-ny} \sin nx$$

by $u_1(x,0) = f(x)$, $C_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx dx$. So

$$u_1(x,t) = \sum_{n=1}^{\infty} (\frac{2}{\pi}\int_0^\pi f(x)\sin nx dx) e^{-ny} \sin nx$$

The second BVP is:

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0, 0 < x < \pi, y > 0$$

$$u_2(0,y) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{if } x \geq 1 \end{cases}, u_2(\pi,y) = e^{-y}$$

$$u_2(x,0) = 0$$

This is a nonhomogeneous Dirichlet boundary condition for the infinite direction y, so use Fourier sine transform with respect to y, we get $\mathcal{F}_s\{u\} = C_1(\alpha)\cosh \alpha x + C_2(\alpha)\sinh \alpha x$, by boundary conditions $C_1(\alpha) = \mathcal{F}_s\{u_2(0,y)\} = \frac{1-\cos\alpha}{\alpha}$. $C_1(\alpha)\cosh \alpha\pi + C_2(\alpha)\sinh \alpha\pi = \mathcal{F}_s\{u_2(\pi,y)\} = \frac{\alpha}{1+\alpha^2}$, so

$$C_2(\alpha) = \frac{1}{\sinh \alpha\pi}[\frac{\alpha}{1+\alpha^2} - \frac{1-\cos\alpha}{\alpha}\cosh \alpha\pi]$$

$$\mathcal{F}_s\{u_2\} = \frac{1-\cos\alpha}{\alpha}\cosh \alpha x + \frac{1}{\sinh \alpha\pi}[\frac{\alpha}{1+\alpha^2} - \frac{1-\cos\alpha}{\alpha}\cosh \alpha\pi]\sinh \alpha x$$

$$u_2(x,y) = \frac{2}{\pi}\int_0^\infty [\frac{1-\cos\alpha}{\alpha}\cosh \alpha x + \frac{1}{\sinh \alpha\pi}(\frac{\alpha}{1+\alpha^2} - \frac{1-\cos\alpha}{\alpha}\cosh \alpha\pi)\sinh \alpha x] \sin \alpha y d\alpha$$

In combination, the solution to the original BVP is

$$\begin{aligned} u(x,y) = u_1(x,y) + u_2(x,y) &= \sum_{n=1}^{\infty} (\frac{2}{\pi}\int_0^\pi f(x)\sin nx dx) e^{-ny} \sin nx + \\ & \frac{2}{\pi}\int_0^\infty [\frac{1-\cos\alpha}{\alpha}\cosh \alpha x + \frac{1}{\sinh \alpha\pi}(\frac{\alpha}{1+\alpha^2} - \frac{1-\cos\alpha}{\alpha}\cosh \alpha\pi)\sinh \alpha x] \sin \alpha y d\alpha \end{aligned}$$

□

Fall 11, #7

Proof. Just look for two constant c_2 and c_3 such that the combination of the boundary conditions with coefficients c_2 and c_3 for (BC2) and (BC3) gives the boundary conditions of (BC1). One needs -1 for (BC2) to fit for $u(1, y)$, need 2 for (BC3) to fit for $u(x, 1)$. So it is $2u_3 - u_2$. □

Fall 11, #13

Proof. Just solve the associated standard wave equation, then add back the given special solution $\cos x$. For (a), you will stop at the infinite sum form with undetermined constant coefficients, since initial conditions are not given there. In (b) you are able to determine those constant coefficients since the initial conditions are given. □

Spring 11, #8

Proof. This is not any equation we have learned, but you can still study it. From $\frac{\partial^2 u}{\partial x^2} = 0$ we know $u(x, y) = p(y)x + q(y)$. Then by $\frac{\partial^2 u}{\partial y^2} = 0$, we have $p''(y)x + q''(y) = 0$, so $p(y) = ay + b$, $q(y) = cy + d$. Therefore the solution is $u(x, y) = Axy + Bx + Cy + D$. Then by the boundary conditions you will have four equations to determine the A, B, C, D. They are $A = 0, B = 3, C = -2, D = 0$. So $u(x, y) = 3x - 2y$, $u(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$. □